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Asymptotic Behavior of the Drift-Diffusion Semiconductor Equations

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In this paper, we continue our study on the asymptotic behavior of the drift-diffusion model for semiconductor devices. We assume the mobilities are constants, and show in this case the dynamical system has a compact, connected, maximal attractor that attracts sets that are bounded in terms of the L^∞ norm. We then prove the differentiability of the semigroup defined by the solution map, and give an upper bound for the Hausdorff dimension of the attractor. © 1995 Academic Press, Inc.

1. INTRODUCTION

For the time-dependent drift-diffusion model for semiconductor devices, we have shown the existence of global solutions in [4] under certain assumptions on the mobilities. It is also shown that the system possesses an absorbing set with respect to the L^∞ norm. In this paper, we assume the mobilities are constants, which is a special case of the mobility model assumed in [4]. In this case the global solution is unique. The system of equations for the electron and hole densities (n, p) is as follows.

$$\frac{\partial n}{\partial t} = \mu_1 \nabla \cdot (\nabla n - n \nabla u) - Q(n, p)(np - 1) + g \qquad \text{in } \Omega, \quad t > 0, \quad (1.1)$$

$$\frac{\partial p}{\partial t} = \mu_2 \nabla \cdot (\nabla p + p \nabla u) - Q(n, p)(np - 1) + g \qquad \text{in } \Omega, \quad t > 0, \quad (1.2)$$

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where u is the electrostatic potential satisfying the Poisson's equations

$$\varepsilon\,\Delta u = n - p - D \qquad \text{in } \Omega, \quad t > 0. \tag{1.3}$$

The boundary conditions are ($\partial\Omega = \Sigma_D \cup \Sigma_N$)

$$(n, p, u) = (\bar{n}, \bar{p}, \bar{u}) \text{ on } \Sigma_D \qquad \text{and} \qquad \frac{\partial n}{\partial \nu} = \frac{\partial p}{\partial \nu} = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Sigma_N, \quad t > 0 \tag{1.4}$$

and initial conditions for $(n, p) = (n_0, p_0)$ are prescribed. The mobilities μ_1 and μ_2, and the permitivity ε are constants. For the physical meanings of the equations and other parameters, we refer to [4] and references therein.

The study of asymptotic properties of the semiconductor equations is quite limited in the existing literature. Mock in [11] first proves that the solution of the time-dependent problem decays exponentially into the solution of the stationary equations under the conditions that the doping profile is constant and the boundary conditions are Nuemann type only. Gajewski [5] and Gajewski and Gröger [6] then extend such asymptotic properties to more general models by assuming a condition (among others) on the boundary data that guarantees the unique solvability of the stationary problem. Although there is no rigorous analytical proof yet, the existence of multiple solutions to the stationary problem has been shown numerically for a one-dimensional case (see, e.g., [12]), and such a phenomenon is expected from the physics view point.

In this paper, without assuming the uniqueness of the stationary problem, we study the time-dependent problem as a dynamical system by applying the general theory of infinite-dimensional dynamical systems (see, e.g., [1, 13]). Our previous study [4] of the global solutions and their L^∞ bounds is the basis of this study here. We first construct an attractor of the dynamical system by using the absorbing set and the global L^∞ bounds for the weak solutions that we have established in [4]. It is also shown that the attractor is compact, connected, and maximal, and it attracts sets that are bounded in terms of the L^∞ norm. The semigroup defined by the solution map is singled valued, and we prove its differentiability by giving the differential. We then show that the Hausdorff dimension of the attractor is finite, and give an upper bound of the dimension.

We remark that existence of $H^1 \cap L^\infty$ weak solutions of the stationary problem has been established in [3] under similar assumptions of this paper. However, at this point, no connection can be made yet between the attractor and the set of all fixed points of the semigroup (i.e., the set of solutions to the stationary problem.)

For most of the arguments in this paper, we follow the framework of general theory of dynamical systems in [13], but we note that it also might be possible to use other approaches, such as these in [9].

We first list our hypotheses on the problem.

(H1) Ω is an open, bounded domain in R^l ($l=1$, 2 or 3) of class $C^{0,1}$, and $\partial\Omega = \Sigma_D \cup \Sigma_N$ with Σ_D being relatively closed.

(H2) D and g are in $L^\infty(\Omega)$ with $g \geqslant 0$ a.e. in Ω.

(H3) The boundary data $\bar{u}$, $\bar{n}$ and $\bar{p}$ are in $L^\infty(\Omega) \cap H^1(\Omega)$ with $\bar{n}$ and $\bar{p}$ positive.

(H4) The initial data n_0 and p_0 are in $L^\infty(\Omega)$ with $n_0(x), p_0(x) \geqslant k_0$ a.e. in Ω for some positive constant k_0. For compatibility, $u_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ is assumed to be the weak solution to

$$\varepsilon\,\Delta u_0 = n_0 - p_0 - D \qquad \text{in } \Omega$$

with the same boundary conditions (1.4) for u.

(H5) The permitivity ε, mobilities μ_1 and μ_2 are positive constants. Set $\underline{\mu} = \min\{\mu_1, \mu_2\}$ and $\bar{\mu} = \max\{\mu_1, \mu_2\}$.

(H6) The function $Q(n, p)$ is differentiable and

$$0 \leqslant Q(n,p) \leqslant \bar{Q}, \qquad |Q_1(n,p)| \leqslant \bar{Q}, \qquad |Q_2(n,p)| \leqslant \bar{Q} \qquad \text{a.e. in } R_+^2$$

where Q_1 and Q_2 are the two partial derivatives of Q.

We define

$$Y = \{\phi\colon \phi \in H^1(\Omega) \text{ and } \phi = 0 \text{ on } \Sigma_D\}$$

and denote the usual $L^2(\Omega)$ inner product as $\langle \phi, \psi \rangle$. For $1 \leqslant r \leqslant \infty$, we denote the $L^r(\Omega)$ norm by $|\phi|_{r,\Omega}$ or simply $|\phi|_r$. Y is equipped with the H^1 equivalent norm $|\nabla \cdot|_2$, and by Poincaré inequality there is a constant $\alpha_r > 0$ so that

$$|\phi|_r \leqslant \alpha_r\, |\nabla\phi|_2 \qquad \text{for all} \quad \phi \in Y, \tag{1.5}$$

where $2 \leqslant r \leqslant 6$ for $l = 1$, 2, 3.

The system of Eqs. (1.1), (1.2), and (1.3) with boundary conditions (1.4) can be formulated in the weak form: $(n, p) \in (\bar{n}, \bar{p}) + Y^2$ satisfies

$$\left\langle \frac{\partial n}{\partial t}, \phi \right\rangle_{Y^* \times Y} + \mu_1 \langle \nabla n - n\,\nabla u, \nabla\phi \rangle + \langle Q(n,p)(np-1) - g, \phi \rangle = 0, \quad t > 0, \tag{1.6}$$

$$\left\langle \frac{\partial p}{\partial t}, \psi \right\rangle_{Y^* \times Y} + \mu_2 \langle \nabla p + p\,\nabla u, \nabla\psi \rangle + \langle Q(n,p)(np-1) - g, \psi \rangle = 0, \quad t > 0 \tag{1.7}$$

for all $(\phi, \psi) \in Y^2$, where the potential $u \in \bar{u} + Y$ satisfies

$$\varepsilon\langle \nabla u, \nabla \zeta \rangle + \langle n - p - D, \zeta \rangle = 0 \tag{1.8}$$

for all $\zeta \in Y$, $t > 0$.

We recapitulate some results obtained in [4] that will be used in this paper. Note that the hypotheses listed above are a special case of these in [4].

(a) For the system (1.6)–(1.7), there is a unique solution

$$(n, p) \in (\bar{n}, \bar{p}) + (L^2_{\text{loc}}(0, \infty; Y) \cap H^1_{\text{loc}}(0, \infty; Y^*))^2$$

and it has the uniform L^∞ bounds:

$$e^{-\bar{d}} \leqslant n(x, t), \quad p(x, t) \leqslant e^{c} \qquad \text{a.e.} \quad x \in \Omega, \quad t > 0 \tag{1.9}$$

where the bounds depend on the initial data (n_0, p_0) and boundary data $(\bar{n}, \bar{p}, \bar{u})$. Moreover, the unique potential u is in $L^\infty([0, \infty) \times \Omega)$. See [4, Theorem 3.1].

(b) The L^∞ bounds for the solution (n, p) with initial condition (n_0, p_0) can be refined as

$$-(d_0 e^{-\omega_2 t} + d) \leqslant \ln n(x, t), \quad \ln p(x, t) \leqslant \ln(C(C_0 e^{-\omega_1 t} + 1)) \tag{1.10}$$

where the constants C and d are independent of the initial condition (n_0, p_0), and the other constants may depend on (n_0, p_0). See [4, Theorem 4.2].

(c) Consider the system as a dynamical system for (n, p) acting on the set of admissible initial data $(n_0, p_0) \in H_{\text{ad}}$ defined by

$$H_{\text{ad}} \equiv \{(n, p) \in L^2(\Omega)^2\colon n, p > 0 \text{ and } \ln n, \ln p \in L^\infty(\Omega)\}. \tag{1.11}$$

Then the solution map defines a single valued semigroup $S(t)(n_0, p_0) = (n(t), p(t))$ on H_{ad} with the L^2-topology. The system has an absorbing set

$$\mathscr{B} = \{(n, p) \in H_{\text{ad}}\colon -(1 + d) \leqslant \ln n, \ln p \leqslant \ln(1 + C)\} \tag{1.12}$$

that absorbs sets that are bounded in terms of the L^∞ norm.

Specific choices for these constants are given in [4]. In particular, if dependent on the initial data $(n_0, p_0) \in H_{\text{ad}}$, these constants only depend on the L^∞-norm of $(\ln n_0, \ln p_0)$ in a regular manner.

2. Existence of the Maximal Attractor

In this section, we study the existence of an attractor for the dynamical system (1.6)–(1.8). The solution map defines a single valued semigroup

$S(t)(n_0, p_0) = (n(t), p(t))$ from H_{ad} into itself (H_{ad} is defined in (1.11).) We equip H_{ad} with the L^2 topology to become a metric space. In the following we will show that this dynamical system has an attractor $\mathscr{A}$ in H_{ad}. By definition (see, e.g., [13]), $\mathscr{A}$ is invariant, and attracts every point of a neighborhood of $\mathscr{A}$ in H_{ad}. In fact, we will show that $\mathscr{A}$ is also compact, connected, attracts sets in H_{ad} that are bounded in terms of the L^∞ norm, and is the maximal of its kind.

First we show the semigroup $S(t)$ is continuous on H_{ad} for each $t > 0$.

LEMMA 2.1. *Suppose $(n(t), p(t))$ and $(\hat{n}(t), \hat{p}(t))$ are solutions to the system corresponding to the initial conditions (n_0, p_0) and $(\hat{n}_0, \hat{p}_0) \in H_{\mathrm{ad}}$, respectively. Then*

$$|n(t) - \hat{n}(t)|_2^2 + |p(t) - \hat{p}(t)|_2^2 \leqslant e^{Mt}(|n_0 - \hat{n}_0|_2^2 + |p_0 - \hat{p}_0|_2^2)$$

for each $t > 0$, where the constant M depends on the initial conditions through the L^∞ bounds of the solutions. Therefore, $S(t)$ is continuous on H_{ad}.

Proof. From (1.6) for both n and $\hat{n}$,

$$\begin{aligned}\langle n_t - \hat{n}_t, \phi\rangle_{Y^* \times Y} &+ \mu_1 \langle (\nabla(n - \hat{n}) - n\,\nabla u + \hat{n}\,\nabla \hat{u}), \nabla\phi\rangle \\ &+ \langle Q(np - 1) - \hat{Q}(\hat{n}\hat{p} - 1), \phi\rangle = 0.\end{aligned}$$

Set $\phi = n - \hat{n}$ to obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}|n - \hat{n}|_2^2\right) + \mu_1\,|\nabla(n - \hat{n})|_2^2 \\
&\quad = \mu_1 \left\langle \nabla u, \nabla\left(\frac{1}{2}(n - \hat{n})^2\right)\right\rangle + \mu_1 \langle \hat{n}\nabla(u - \hat{u}), \nabla(n - \hat{n})\rangle \\
&\qquad - \langle Q(np - 1) - \hat{Q}(\hat{n}\hat{p} - 1), n - \hat{n}\rangle \\
&\quad \leqslant -\frac{\mu_1}{\varepsilon}\left\langle n - p - D, \frac{1}{2}(n - \hat{n})^2\right\rangle \quad \left(\text{from (1.8) for } u \text{ with } \zeta = \frac{1}{2}(n - \hat{n})^2\right) \\
&\qquad + \mu_1\,|\hat{n}|_\infty\,|\nabla(u - \hat{u})|_2\,|\nabla(n - \hat{n})|_2 + M_1(|n - \hat{n}|_2 + |p - \hat{p}|_2)\,|n - \hat{n}|_2 \\
&\qquad\quad (\text{by (H6) and the } L^\infty \text{ bounds for } n \text{ and } \hat{n}) \\
&\quad \leqslant \left(\frac{\mu_1}{2\varepsilon}(|n|_\infty + |p|_\infty + |D|_\infty) + \frac{3}{2}M_1\right)|n - \hat{n}|_2^2 + \frac{1}{2}M_1\,|p - \hat{p}|_2^2 \\
&\qquad + \frac{\mu_1}{2}|\hat{n}|_\infty^2\,|\nabla(u - \hat{u})|_2^2 + \frac{\mu_1}{2}|\nabla(n - \hat{n})|_2^2
\end{aligned}$$

where $M_1 = \bar{Q}\,(e^c + 1)^2$. From (1.8) for both u and $\hat{u}$,

$$\langle \varepsilon\nabla(u-\hat{u}), \nabla\zeta\rangle + \langle (n-\hat{n})-(p-\hat{p}), \zeta\rangle = 0,$$

and by setting $\zeta = u - \hat{u}$ we can obtain

$$|\nabla(u-\hat{u})|_2^2 \leqslant \frac{2\alpha_2^2}{\varepsilon^2}(|n-\hat{n}|_2^2 + |p-\hat{p}|_2^2).$$

Hence,

$$\frac{d}{dt}\left(\frac{1}{2}|n-\hat{n}|_2^2\right) + \frac{\mu_1}{2}|\nabla(n-\hat{n})|_2^2 \leqslant M_2\,|n-\hat{n}|_2^2 + M_3\,|p-\hat{p}|_2^2$$

where

$$M_2 = \frac{\bar{\mu}}{2\varepsilon}\left(2e^c + |D|_\infty + \frac{3}{2}M_1\right) + \frac{\bar{\mu}\alpha_2^2}{\varepsilon^2}e^{2c} \qquad \text{and} \qquad M_3 = \frac{1}{2}M_1 + \frac{\bar{\mu}\alpha_2^2}{\varepsilon^2}e^{2c}.$$

Similar estimate can be obtained for $p - \hat{p}$:

$$\frac{d}{dt}\left(\frac{1}{2}|p-\hat{p}|_2^2\right) + \frac{\mu_2}{2}|\nabla(p-\hat{p})|_2^2 \leqslant M_2\,|p-\hat{p}|_2^2 + M_3\,|n-\hat{n}|_2^2.$$

Therefore,

$$\frac{d}{dt}(|n-\hat{n}|_2^2 + |p-\hat{p}|_2^2) \leqslant 2(M_2+M_3)(|n-\hat{n}|_2^2 + |p-\hat{p}|_2^2),$$

and the lemma follows with $M = 2(M_2 + M_3)$. ∎

In order to have certain compactness for $S(t)$, we also need some regularity estimates for the solutions with respect to time t.

LEMMA 2.2. *The solution $(n(t), p(t))$ with initial data $(n_0, p_0) \in H_{\mathrm{ad}}$ has the following estimate*:

$$\int_0^t |\nabla(n(s)-\bar{n})|_2^2\,ds, \qquad \int_0^t |\nabla(p(s)-\bar{p})|_2^2\,ds \leqslant M(1+t)$$

for all $t \geqslant 0$, where M is a constant depending on the L^∞-norm of $(\ln n_0, \ln p_0)$.

Proof. Choose $\phi = n(t) - \bar{n}$ in (1.6) to obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}|n-\bar{n}|_2^2\right) + \mu_1 |\nabla(n-\bar{n})|_2^2 \\
&\quad = -\mu_1\langle \nabla\bar{n}, \nabla(n-\bar{n})\rangle + \mu_1\langle n\nabla u, \nabla(n-\bar{n})\rangle - \langle Q(np-1) - g, n-\bar{n}\rangle \\
&\quad \leqslant \mu_1 |\nabla\bar{n}|_2 |\nabla(n-\bar{n})|_2 + \mu_1 e^c |\nabla u|_2 |\nabla(n-\bar{n})|_2 \\
&\qquad + [\bar{Q}(e^{2c}+1)|\Omega|^{1/2} + |g|_2] |n-\bar{n}|_2 \\
&\quad \leqslant \frac{3\mu_1}{4}|\nabla(n-\bar{n})|_2^2 + \mu_1|\nabla\bar{n}|_2^2 + \mu_1 e^{2c}|\nabla u|_2^2 \\
&\qquad + \frac{\alpha_2^2}{\underline{\mu}}[\bar{Q}(e^{2c}+1)|\Omega|^{1/2} + |g|_2]^2.
\end{aligned}$$

Since, from (1.8) with $\zeta = u - \bar{u}$,

$$\begin{aligned}
\varepsilon|\nabla(u-\bar{u})|_2^2 &= -\varepsilon\langle\nabla\bar{u}, \nabla(u-\bar{u})\rangle - \langle n-p-D, u-\bar{u}\rangle \\
&\leqslant \frac{\varepsilon}{4}|\nabla(u-\bar{u})|_2^2 + \varepsilon|\nabla\bar{u}|_2^2 + (2e^c + |D|_\infty)|\Omega|^{1/2}|u-\bar{u}|_2 \\
&\leqslant \frac{\varepsilon}{2}|\nabla(u-\bar{u})|_2^2 + \varepsilon|\nabla\bar{u}|_2^2 + \frac{\alpha_2^2}{\varepsilon}(2e^c+|D|_\infty)^2|\Omega|,
\end{aligned}$$

we have

$$|\nabla u|_2^2 \leqslant 2|\nabla(u-\bar{u})|_2^2 + 2|\nabla\bar{u}|_2^2 \leqslant 6|\nabla\bar{u}|_2^2 + \frac{4\alpha_2^2}{\varepsilon^2}(2e^c + |D|_\infty)^2|\Omega| \equiv M_4$$

for all t. Therefore,

$$\frac{d}{dt}\left(\frac{1}{2}|n-\bar{n}|_2^2\right) + \frac{\mu_1}{4}|\nabla(n-\bar{n})|_2^2 \leqslant M_5$$

where

$$M_5 = \bar{\mu}(|\nabla\bar{n}|_2^2 + |\nabla\bar{p}|_2^2) + \bar{\mu}e^{2c}M_4 + \frac{\alpha_2^2}{\underline{\mu}}[\bar{Q}(e^{2c}+1)|\Omega|^{1/2} + |g|_2]^2.$$

Hence,

$$\frac{\mu_1}{4}\int_0^t |\nabla(n(s)-\bar{n})|_2^2\, ds \leqslant \frac{1}{2}|n_0 - \bar{n}|_2^2 + M_5 t$$

and the lemma follows for $n - \bar{n}$. The same estimate can be obtained for $p - \bar{p}$.

LEMMA 2.3. *For a fixed $T > 0$,*

$$t^2 \left(\left| \frac{\partial n(t)}{\partial t} \right|_2^2 + \left| \frac{\partial p(t)}{\partial t} \right|_2^2 \right) + \int_0^t s^2 \left(\left| \nabla \frac{\partial n(s)}{\partial t} \right|_2^2 + \left| \nabla \frac{\partial p(s)}{\partial t} \right|_2^2 \right) ds \leqslant M_T$$

for $t \in [0, T]$, where the constant M_T depends on T.

Proof. We prove this lemma by estimating the corresponding difference quotients. For a function $f(t)$, define the difference quotient

$$f_h(t) = \frac{f(t+h) - f(t)}{h}$$

for $h \neq 0$. From (1.6) with $t = s$, $t = s + h$ and $\phi = n_h(s)$,

$$\begin{aligned}
&\langle n_h'(s), n_h(s) \rangle_{Y^* \times Y} + \mu_1 \langle \nabla n_h(s), \nabla n_h(s) \rangle \\
&\quad = \mu_1 \left\langle \frac{n(s+h)\, \nabla u(s+h) - n(s)\, \nabla u(s)}{h}, \nabla n_h(s) \right\rangle \\
&\qquad - \left\langle \frac{1}{h} Q(n, p)(np - 1) \Big|_s^{s+h}, n_h(s) \right\rangle \\
&\quad \leqslant \mu_1 \left\langle \nabla u(s+h), \nabla \frac{1}{2} n_h(s)^2 \right\rangle + \mu_1 \langle n(s)\, \nabla u_h(s), \nabla n_h(s) \rangle \\
&\qquad + \int_\Omega M_1 (|n_h(s)| + |p_h(s)|)\, |n_h(s)|\, dx \\
&\qquad \text{(by (H6) and the } L^\infty \text{ bounds for } (n, p)) \\
&\quad \leqslant \frac{\mu_1}{\varepsilon} \left\langle n(s+h) - p(s+h) - D, \frac{1}{2} n_h(s)^2 \right\rangle \quad \text{(by (1.8) with } \zeta = n_h(s)^2) \\
&\qquad + \mu_1 e^{\bar{c}}\, |\nabla u_h(s)|_2\, |\nabla n_h(s)|_2 + \frac{3}{2} M_1\, |n_h|_2^2 + \frac{1}{2} M_1\, |p_h|_2^2 \\
&\quad \leqslant \left(\frac{\mu_1}{2\varepsilon} (2e^{\bar{c}} + |D|_\infty) + \frac{3}{2} M_1 \right) |n_h(s)|_2^2 + \frac{1}{2} M_1\, |p_h|_2^2 \\
&\qquad + \frac{\mu_1}{2} |\nabla n_h(s)|_2^2 + \frac{\mu_1}{2} e^{2\bar{c}}\, |\nabla u_h(s)|_2^2.
\end{aligned}$$

From (1.8),

$$\langle \varepsilon\, \nabla u_h(s), \nabla u_h(s) \rangle + \langle n_h(s) - p_h(s), u_h(s) \rangle = 0,$$

and hence

$$|\nabla u_h(s)|_2^2 \leqslant \frac{2\alpha_2^2}{\varepsilon^2}\,(|n_h(s)|_2^2 + |p_h(s)|_2^2).$$

Therefore,

$$\langle n_h'(s), n_h(s)\rangle_{Y^*\times Y} + \frac{\mu_1}{2}\langle \nabla n_h(s), \nabla n_h(s)\rangle \leqslant M_2\,|n_h(s)|_2^2 + M_3\,|p_h(s)|_2^2.$$

Note that,

$$\frac{t^2}{2}\,|n_h(t)|_2^2 = \int_0^t s\,|n_h(s)|_2^2\,ds + \int_0^t s^2\langle n_h'(s), n_h(s)\rangle_{Y^*\times Y}\,ds.$$

Hence

$$\frac{t^2}{2}\,|n_h(t)|_2^2 \leqslant \int_0^t s\,|n_h(s)|_2^2\,ds - \frac{\mu_1}{2}\int_0^t s^2\,|\nabla n_h(s)|_2^2\,ds$$
$$+ \int_0^t s^2(M_2\,|n_h(s)|_2^2 + M_3\,|p_h(s)|_2^2)\,ds,$$

that is,

$$\frac{t^2}{2}\,|n_h(t)|_2^2 + \frac{\mu_1}{2}\int_0^t s^2\,|\nabla n_h(s)|_2^2\,ds$$
$$\leqslant \int_0^t [(s + M_2 s^2)\,|n_h(s)|_2^2 + M_3 s^2\,|p_h(s)|_2^2]\,ds.$$

Similarly, we have

$$\frac{t^2}{2}\,|p_h(t)|_2^2 + \frac{\mu_2}{2}\int_0^t s^2\,|\nabla p_h(s)|_2^2\,ds$$
$$\leqslant \int_0^t [(s + M_2 s^2)\,|p_h(s)|_2^2 + M_3 s^2\,|n_h(s)|_2^2]\,ds.$$

Hence

$$\frac{t^2}{2}\,(|n_h(t)|_2^2 + |p_h(t)|_2^2) + \frac{1}{2}\int_0^t s^2(\mu_1\,|\nabla n_h(s)|_2^2 + \mu_2\,|\nabla p_h(s)|_2^2)\,ds$$
$$\leqslant \int_0^t (1 + (M_2 + M_3)\,s)\,s(|n_h(s)|_2^2 + |p_h(s)|_2^2)\,ds.$$

Note further that

$$|n_h(s)|_2^2 = \langle n_h(s), n_h(s) \rangle \leqslant |n_h(s)|_{Y^*} |\nabla n_h(s)|_2$$

and similarly for p_h. Therefore from the above estimate we obtain

$$\frac{t^2}{2}(|n_h(t)|_2^2 + |p_h(t)|_2^2) + \frac{1}{4}\int_0^t s^2(\mu_1 |\nabla n_h(s)|_2^2 + \mu_2 |\nabla p_h(s)|_2^2)\, ds$$

$$\leqslant \int_0^t \frac{1}{\underline{\mu}}(1 + (M_2 + M_3)\, s)^2 (|n_h(s)|_{Y^*}^2 + |p_h(s)|_{Y^*}^2)\, ds.$$

Since we have

$$\frac{\partial n}{\partial t}, \frac{\partial p}{\partial t} \in L^2(0, T; Y^*),$$

the estimate in the lemma follows by using the relations between difference quotients and derivatives (see, e.g., [7]). ∎

Recall that a set B in H_{ad} is said to be L^∞-bounded if there exists a constant K_B so that

$$|\ln n|_\infty, |\ln p|_\infty \leqslant K_B \qquad \text{for all} \quad (n, p) \in B.$$

LEMMA 2.4. *For a given L^∞-bounded set B in H_{ad}, $S(t)\, B$ is precompact in H_{ad} for each $t > 0$.*

Proof. Since

$$\frac{t^2}{2} |\nabla n(t)|_2^2 = \int_0^t s\, |\nabla n(s)|_2^2 + \int_0^t s^2 \langle \nabla n'(s), \nabla n(s) \rangle,$$

from Lemmas 2.2 and 2.3, the solution with initial condition $(n_0, p_0) \in B$ has the following bound:

$$|\nabla n(t)|_2^2, |\nabla p(t)|_2^2 \leqslant M_T.$$

Moreover, since the bound M_T depends only on the L^∞ norm of (n_0, p_0), it can be chosen to be uniform for all the solutions whose initial data are from the L^∞-bounded set B. Hence $S(t)\, B$ is a bounded set in $H^1(\Omega)^2$. By the compact embedding from H^1 to L^2, the set $S(t)\, B$ is precompact in $L^2(\Omega)^2$. To conclude that $S(t)\, B$ is also precompact in H_{ad}, we need to show that $\overline{S(t)\, B}$ is contained in H_{ad}, where the closure is taken in the L^2-topology. But this is a direct result of the fact that every L^2 convergent sequence has an a.e. pointwise convergent subsequence and the fact that $S(t)\, B$ is L^∞-bounded due to (1.9). ∎

THEOREM 2.5. *The dynamical system on H_{ad} has a compact attractor $\mathscr{A}$ that attracts L^∞-bounded sets in H_{ad}. Moreover, $\mathscr{A}$ is L^∞-bounded, and it is the largest among such attractors. $\mathscr{A}$ is also connected, and is contained in $H^1(\Omega)^2$.*

Proof. For the set $\mathscr{B}$ constructed in (1.12), let

$$\mathscr{A} = \omega(\mathscr{B}) = \bigcap_{s \geqslant 0} \overline{\bigcup_{t \geqslant s} S(t)\, \mathscr{B}},$$

the ω-limit set in H_{ad}, where the closure is taken in the L^2-topology. Clearly $\mathscr{A}$ is L^∞-bounded due to the global bounds (1.9) of the solutions.

From (1.10), we can see that there is t_0 such that $S(t)\,\mathscr{B} \subset \mathscr{B}$ for $t \geqslant t_0$. Hence $\bigcup_{t \geqslant t_0} S(t)\,\mathscr{B}$ is L^∞-bounded. Therefore $\bigcup_{t \geqslant 2t_0} S(t)\,\mathscr{B} = S(t_0) \bigcup_{t \geqslant t_0} S(t)\,\mathscr{B}$ is precompact in H_{ad} by Lemma 2.4. We then apply [13, Lemma 1.1] to conclude that $\mathscr{A} = \omega(\mathscr{B})$ is nonempty, compact and invariant.

We claim that $\mathscr{A}$ attracts every L^∞-bounded set in H_{ad}. If otherwise, there is an L^∞-bounded set $B_0 \subset H_{\mathrm{ad}}$ such that $dist(S(t)\, B_0, \mathscr{A})$ does not tend to 0 as $t \to \infty$. (Here the *dist* is by the L^2-topology.) That is, there exist $\delta > 0$, $t_k \to \infty$ and $x_k \in B_0$ such that

$$dist(S(t_k)\, x_k, \mathscr{A}) \geqslant \delta > 0 \tag{2.1}$$

for all $k = 1, 2, \ldots$. Again by (1.10), there is t_0 such that $S(t)\, B_0 \subset \mathscr{B}$ when $t \geqslant t_0$. Without loss of generality, we assume $t_k \geqslant 2t_0$ for all $k = 1, 2, \ldots$. Then the sequence $S(t_k - t_0)\, x_k$ is contained in $\mathscr{B}$, and $S(t_k)\, x_k = S(t_0)\, S(t_k - t_0)\, x_k$ is precompact in H_{ad} due to Lemma 2.4. Thus $S(t_k)\, x_k$ has a convergent subsequence $S(t_{k_j})\, x_{k_j} \to y_0 \in H_{\mathrm{ad}}$ as $j \to \infty$. Note that $y_0 = \lim_{j \to \infty} S(t_{k_j} - t_0)\, y_j$ with $y_j = S(t_0)\, x_{k_j} \in \mathscr{B}$. Hence $y_0 \in \mathscr{A}$ by the definition of ω-limit set. This contradicts (2.1).

$\mathscr{A}$ is the maximal attractor. Suppose there is another L^∞-bounded attractor $\mathscr{A}_1 \supset \mathscr{A}$. Since $\mathscr{B}$ absorbs L^∞-bounded set, $S(t)\, \mathscr{A}_1 \subset \mathscr{B}$ for t large enough. Hence $\mathscr{A}_1 = S(t)\, \mathscr{A}_1 \subset \mathscr{B}$, and $\mathscr{A}_1 = \omega(\mathscr{A}_1) \subset \omega(\mathscr{B}) = \mathscr{A}$. Therefore, $\mathscr{A}_1 = \mathscr{A}$.

Finally, $\mathscr{A}$ is also connected. We show this by contradiction. Suppose $\mathscr{A}$ is not connected. Then there are open sets O_1 and O_2 in H_{ad} such that $O_1 \cap O_2 = \varnothing$ and $\mathscr{A} \subset O_1 \cup O_2$. Let $K = \operatorname{co} \mathscr{A}$ be the convex hull of $\mathscr{A}$. Then K is connected. $K \subset H_{\mathrm{ad}}$ because H_{ad} is convex, and K is L^∞-bounded since $\mathscr{A}$ is. Note that $\mathscr{A} = S(t)\, \mathscr{A} \subset S(t)\, K$ and $S(t)\, K$ is also connected since $S(t)$ is continuous for each $t > 0$ (Lemma 2.1). Therefore $O_1 \cup O_2$ does not cover $S(t)\, K$ for each $t > 0$. That is, there are $x_k \in K$ such that $S(k)\, x_k \notin O_1 \cup O_2$ for all $k = 1, 2, \ldots$. Since it is easy to see that $S(k)\, x_k$ is precompact in H_{ad} for k large enough (by Lemma 2.4), there exists a subsequence of $S(k)x_k$ that converges to some $y \in H_{ad}$. Clearly $y \notin O_1 \cup O_2$. On the other hand, since $\mathscr{A}$ attracts K, y must be in $\mathscr{A}$. This is a contradiction.

Since for $(n_0, p_0) \in H_{ad}$ and $t > 0$, $S(t)(n_0, p_0) \in H^1(\Omega)^2$, and $\mathscr{A} = S(t)\mathscr{A}$, clearly $\mathscr{A}$ is contained in $H^1(\Omega)^2$. ∎

We remark that our proof here is similar to that of [13, Theorem 1.1], which cannot be applied directly because the set $\mathscr{B}$ absorbs only sets that are bounded with respect to the L^∞ norm, a topology stronger than that of H_{ad} (the L^2-topology).

COROLLARY 2.6. *If (n, p) is the solution to the system (1.6)–(1.7) with the initial condition $(n_0, p_0) \in \mathscr{A}$, i.e. $(n(t), p(t)) = S(t)(n_0, p_0)$, then $(n, p) \in \mathscr{A}$ for all $t \geq 0$ and*

$$e^{-(1+d)} \leq n(x, t),\ p(x, t) \leq 1 + C \qquad a.e.\quad x \in \Omega,\quad t \geq 0$$

where the constants d and C are independent of $(n_0, p_0) \in \mathscr{A}$ and are the same constants as in (1.12).

Proof. This is obvious from the fact that $S(t)\mathscr{A} = \mathscr{A}$ and $\mathscr{A} \subset \mathscr{B}$.

We will need this result for estimating the dimension of $\mathscr{A}$ in Section 4.

3. THE LINEARIZED SYSTEM AND DIFFERENTIABILITY OF THE SEMIGROUP

In this section, we analyze the first variation system of (1.6)–(1.7) (i.e. the linearized system) and the differentiability of the semigroup. These results are also needed for the study of the Hausdorff dimension of the attractor $\mathscr{A}$ in the next section.

In the following, we need a regularity assumption for the Laplacian operator with the mixed boundary conditions:

(H7) (i) For $f \in L^2(\Omega)$, the weak solution u to the equation

$$\Delta u = f \text{ with } u = 0 \text{ on } \Sigma_D \qquad \text{and} \qquad \frac{\partial u}{\partial \nu} = 0 \text{ on } \Sigma_N$$

is in $W^{1,4}(\Omega)$ and there holds the estimate

$$|u|_{W^{1,4}(\Omega)} \leq c\,|f|_2$$

for some constant c independent of f.

(ii) The boundary data $\bar{u}$ in (1.4) is in $L^\infty(\Omega) \cap H^2(\Omega)$.

Note that in the one-dimensional case (H7i) always holds, and it is also satisfied for some two or three-dimensional cases. For instance, in the two-dimensional case, this assumption holds when Ω is a rectangle and Dirichlet conditions are given on the two opposite sides of the rectangle, or more generally when Ω is a curvilinear polygon with angles determined by Σ_D and Σ_N less than $\pi/2$ (see [8]). In general, it is satisfied when Σ_D is both relatively closed and open in $\partial\Omega$ (see [14]).

The linearized system of (1.1)–(1.2) at the solution $(n, p) = S(t)(n_0, p_0)$ (with $(n_0, p_0) \in H_{\text{ad}}$ and u given by (1.3)) can be formally written for (N, P) as

$$\frac{\partial N}{\partial t} = \mu_1 \nabla \cdot (\nabla N - N \nabla u - n \nabla U) - R_1 N - R_2 P \tag{3.1}$$

$$\frac{\partial P}{\partial t} = \mu_2 \nabla \cdot (\nabla P + P \nabla u + p \nabla U) - R_1 N - R_2 P \tag{3.2}$$

where U satisfies

$$\varepsilon \Delta U = N - P \tag{3.3}$$

with homogeneous boundary conditions

$$N = P = U = 0 \text{ on } \Sigma_D \qquad \text{and} \qquad \frac{\partial N}{\partial \nu} = \frac{\partial P}{\partial \nu} = \frac{\partial U}{\partial \nu} = 0 \text{ on } \Sigma_N \tag{3.4}$$

and prescribed initial conditions for $(N, P) = (N_0, P_0)$. Here we denote

$$R_1 = pQ(n, p) + (np - 1)\, Q_1(n, p)$$

and

$$R_2 = nQ(n, p) + (np - 1)\, Q_2(n, p).$$

Because of (H6) and the bounds (1.9), clearly we have $|R_1|, |R_2| \leqslant \bar{Q}(e^c + 1)^2$.

It is usually more convenient to work with the weak form of the system: $(N, P) \in L^2(0, T; Y)^3$ satisfies

$$\langle N_t, \phi \rangle_{Y^* \times Y} + \mu_1 \langle \nabla N - N \nabla u - n \nabla U, \nabla \phi \rangle + \langle R_1 N + R_2 P, \phi \rangle = 0 \tag{3.5}$$

$$\langle P_t, \psi \rangle_{Y^* \times Y} + \mu_2 \langle \nabla P + P \nabla u + p \nabla U, \nabla \psi \rangle + \langle R_1 N + R_2 P, \psi \rangle = 0 \tag{3.6}$$

for all $(\phi, \psi) \in Y^2$, where $U \in Y$ satisfies

$$\varepsilon\langle \nabla U, \nabla\zeta\rangle + \langle N - P, \zeta\rangle = 0 \tag{3.7}$$

for all $\eta \in Y$, $t \geqslant 0$.

We first establish the existence and regularity results for this system.

THEOREM 3.1. *For $T > 0$ fixed, the linear system* (3.5)–(3.6) *for* (N, P) *has a unique weak solution*

$$(N(t), P(t)) \in [L^2(0, T; Y) \cap H^1(0, T; Y^*) \cap C([0, T]; L^2(\Omega))]^2$$

with each initial condition $(N_0, P_0) \in L^2(\Omega)^2$. *Moreover, there exists a constant k such that*

$$|N(t)|_2^2, |P(t)|_2^2 \leqslant e^{kt}(|N_0|_2^2 + |P_0|_2^2) \qquad \textit{for all} \quad t \geqslant 0. \tag{3.8}$$

Proof. We prove the existence result by following an abstract approach to the Cauchy problem (see, e.g., [15, Section 26])

$$\left\langle \frac{\partial Z}{\partial t}, \xi \right\rangle_{\mathscr{Y}^* \times \mathscr{Y}} + \sigma(Z, \xi) = 0 \qquad \text{for all} \quad \xi \in \mathscr{Y} \text{ with } Z(0) = Z_0 \in \mathscr{H} \tag{3.9}$$

where we identify $Z = (N, P)$, $\xi = (\phi, \psi)$, $Z_0 = (N_0, P_0)$, $\mathscr{H} = L^2(\Omega)^2$, $\mathscr{Y} = Y^2$, and the sesquilinear form $\sigma(Z, \xi)$ is defined as

$$\begin{aligned}\sigma(Z, \xi) = {} & \mu_1\langle \nabla N - N\,\nabla u - n\,\nabla U, \nabla\phi\rangle + \mu_2\langle \nabla P + P\,\nabla u + p\,\nabla U, \nabla\psi\rangle \\ & + \langle R_1 N + R_2 P, \phi + \psi\rangle \end{aligned} \tag{3.10}$$

where U is related to (N, P) by (3.7). Note that σ itself depends on t through (n, p, u), the solution to (1.6)–(1.8). To prove the existence result, it suffices to verify the continuity and coercivity of σ:

$$|\sigma(Z, \xi)| \leqslant \gamma\,|Z|_{\mathscr{Y}}\,|\xi|_{\mathscr{Y}}, \qquad \sigma(Z, Z) \geqslant \alpha\,|Z|_{\mathscr{Y}}^2 - \beta\,|Z|_{\mathscr{H}}^2 \tag{3.11}$$

for all $Z, \xi \in \mathscr{Y}$.

For the continuity, we first notice that from (1.3) and (H7), u is in $W^{1,4}(\Omega)$. Hence

$$|\langle N\,\nabla u, \nabla\phi\rangle| \leqslant |N|_4\,|\nabla u|_4\,|\nabla\phi|_2 \leqslant \alpha_4\,|\nabla N|_2\,|u|_{W^{1,4}}\,|\nabla\phi|_2$$

where we use the Poincaré inequality (1.5) for $|N|_4$. By (3.7), we also have

$$|\langle n\,\nabla U, \nabla\phi\rangle| \leqslant e^c\,|\nabla U|_2\,|\nabla\phi|_2 \leqslant \frac{1}{\varepsilon}\,e^c(|N|_2 + |P|_2)\,|\nabla\phi|_2. \tag{3.12}$$

Therefore, we can easily obtain the continuity of σ on $\mathscr{Y} \times \mathscr{Y}$.

For the coercivity, note that $N^2 \in Y$ for $N \in Y \cap L^\infty(\Omega)$, and from (1.8) by setting $\zeta = N^2$,

$$-\langle N \nabla u, \nabla N \rangle = -\frac{1}{2} \langle \nabla u, \nabla N^2 \rangle = \frac{1}{2\varepsilon} \langle n - p - D, N^2 \rangle$$

$$\geqslant -\frac{1}{2\varepsilon} (e^c + |D|_\infty) |N|_2^2$$

since $n > 0$. Similar to (3.12), we have

$$\langle n \nabla U, \nabla N \rangle \geqslant -\frac{1}{\varepsilon} e^c(|N|_2 + |P|_2) |\nabla N|_2 \geqslant -\frac{1}{2} |\nabla N|_2^2 - \frac{1}{\varepsilon^2} e^{2c}(|N|_2^2 + |P|_2^2).$$

With the similar estimates for P, we can obtain, for $N, P \in Y \cap L^\infty(\Omega)$,

$$\begin{aligned}
\sigma(Z, Z) \geqslant{} & \mu_1 |\nabla N|_2^2 + \mu_2 |\nabla P|_2^2 - \frac{1}{2\varepsilon} (e^c + |D|_\infty)(\mu_1 |N|_2^2 + \mu_2 |P|_2^2) \\
& - \frac{1}{2} (\mu_1 |\nabla N|_2^2 + \mu_2 |\nabla P|_2^2) - \frac{1}{\varepsilon^2} e^{2c}(\mu_1 + \mu_2)(|N|_2^2 + |P|_2^2) \\
& - 2\bar{Q}(e^c + 1)^2 (|N|_2^2 + |P|_2^2) \\
\geqslant{} & \frac{1}{2} (\mu_1 |\nabla N|_2^2 + \mu_2 |\nabla P|_2^2) \\
& - \left[\frac{1}{2\varepsilon} \bar{\mu}(e^c + |D|_\infty) + \frac{1}{\varepsilon^2} e^{2c}(\mu_1 + \mu_2) + 2\bar{Q}(e^c + 1)^2 \right] (|N|_2^2 + |P|_2^2).
\end{aligned}$$

That is, σ is coercive for all $Z = (N, P) \in (Y \cap L^\infty(\Omega))^2$. Since $Y \cap L^\infty(\Omega)$ is dense in Y, σ is also coercive on $\mathscr{V}$. The constants α and β in (3.11) are

$$\alpha = \frac{1}{2} \underline{\mu} \qquad \text{and} \qquad \beta = \frac{1}{2\varepsilon} \bar{\mu}(e^c + |D|_\infty) + \frac{2}{\varepsilon^2} \bar{\mu} e^{2c} + 2\bar{Q}(e^c + 1)^2. \tag{3.13}$$

Thus, by applying the existence and regularity results for the abstract Cauchy problem (see, e.g., [15]), we obtain the existence result stated in the theorem.

To obtain (3.8), we set $\xi = Z$ in (3.9) and use the coercivity of the σ to yield

$$\frac{d}{dt} |Z(t)|_{\mathscr{H}}^2 \leqslant k\, |Z(t)|_{\mathscr{H}}^2$$

with $k = 2\beta$. Hence (3.8) follows immediately from the Gronwall inequality. ∎

In fact, the solution is in $L^{\infty}([0, T] \times \Omega)$. See, e.g., [10].

PROPOSITION 3.2. *The solution* (N, P) *to* (3.5)–(3.7) *is in* $L^{\infty}([0, T] \times \Omega)^2$.

We can also obtain an useful estimate for $U(t)$ from (3.7) by using (3.8) and the regularity assumption (H7).

PROPOSITION 3.3. *The solution* $U(t) \in L^2(0, T; Y)$ *to* (3.7) *is in* $W^{1,4}(\Omega)$ *and satisfies*

$$|U(t)|^2_{W^{1,4}} \leqslant Ke^{kt}(|N_0|_2^2 + |P_0|_2^2) \qquad \text{for all} \quad t \geqslant 0. \tag{3.14}$$

We denote the solution map of this linearized system by a linear map on $\mathscr{H}$:

$$L(t, (n_0, p_0)) \colon Z_0 = (N_0, P_0) \in \mathscr{H} \to Z(t) = (N(t), P(t)) \in \mathscr{H}.$$

Next, we show that the semigroup $S(t)$ is differentiable and its Fréchet differential is L.

THEOREM 3.4. *Under the hypotheses* (H1)–(H7), *the semigroup* $S(t)$, *defined by the solution map of* (1.6)–(1.7) *with initial condition* $(n_0, p_0) \in H_{\text{ad}} \subset \mathscr{H}$, *is differentiable in* $\mathscr{H}$, *and its Fréchet differential is given by the linear map L defined by the solution of the first variation system* (3.5)–(3.6).

Proof. Let $z(t) = S(t)\, z_0$ and $\hat{z}(t) = S(t)\, \hat{z}_0$ with $z_0 = (n_0, p_0)$, $\hat{z}_0 = (\hat{n}_0, \hat{p}_0) \in H_{\text{ad}}$. And let $Z(t) = L(t, z_0)(\hat{z}_0 - z_0)$. Assume that (n, p) and $(\hat{n}, \hat{p})$ have the same bounds as in (1.9) by choosing appropriate $\bar{c}$ and $\bar{d}$. We show that

$$|\hat{z}(t) - z(t) - Z(t)|_{\mathscr{H}} = o(|\hat{z}_0 - z_0|_{\mathscr{H}}) \tag{3.15}$$

and thus the differentiability of S in $\mathscr{H}$ with differential L follows immediately. To this end, we first write down the equations for $(\delta n, \delta p) \equiv (\hat{n} - n - N, \hat{p} - p - P)$ from (1.6)–(1.7) and (3.5)–(3.6):

$$\begin{aligned} &\langle \delta n_t, \phi \rangle + \mu_1 \langle \nabla\, \delta n, \nabla \phi \rangle + \mu_1 \langle \delta n\, \nabla u + \hat{n} \nabla\, \delta u \\ &\qquad + (\hat{n} - n) \nabla U, \nabla \phi \rangle + \langle F, \phi \rangle = 0 \end{aligned} \tag{3.16}$$

$$\begin{aligned} &\langle \delta p_t, \psi \rangle + \mu_2 \langle \nabla\, \delta p, \nabla \psi \rangle - \mu_2 \langle \delta p\, \nabla u + \hat{p} \nabla\, \delta u \\ &\qquad + (\hat{p} - p) \nabla U, \nabla \psi \rangle + \langle F, \psi \rangle = 0 \end{aligned} \tag{3.17}$$

and the equation for $\delta u \equiv \hat{u} - u - U$ from (1.8) and (3.7),

$$\varepsilon\langle \nabla \delta u, \nabla \zeta\rangle + \langle \delta n - \delta p, \zeta\rangle = 0 \tag{3.18}$$

for all $(\phi, \psi, \zeta) \in Y^3$, where we have denoted

$$\begin{aligned} F = &= Q(\hat{n}\,\delta p + p\,\delta n + (\hat{n} - n)\,P) \\ &+ (\hat{n}\hat{p} - 1)(\hat{Q} - Q - Q_1 N - Q_2 P) + (\hat{n}\hat{p} - np)(Q_1 N + Q_2 P). \end{aligned}$$

Then we set $\phi = \delta n$ in (3.16) and $\psi = \delta p$ in (3.17), and estimate the terms as follows.

$$\begin{aligned} \langle \delta n\,\nabla u, \nabla\,\delta n\rangle &= \frac{1}{2}\langle \nabla u, \nabla\,\delta n^2\rangle \\ &= -\frac{1}{2\varepsilon}\langle n - p - D, \delta n^2\rangle \\ &\qquad \text{(from (1.8) with } \zeta = \delta n^2 \in Y) \\ &\leqslant \frac{1}{2\varepsilon}(e^c + |D|_\infty)\,|\delta n|_2^2 \quad (\text{since } n > 0). \end{aligned}$$

$$\begin{aligned} \langle \hat{n}\nabla\,\delta u, \nabla\,\delta n\rangle &\leqslant e^c\,|\nabla\,\delta u|_2\,|\nabla\,\delta n|_2 \\ &\leqslant \frac{\alpha_2}{\varepsilon} e^c(|\delta n|_2 + |\delta p|_2)\,|\nabla\,\delta n|_2 \\ &\qquad \text{(using (3.18) with } \zeta = \delta u) \\ &\leqslant \frac{1}{4}|\nabla\,\delta n|_2^2 + \left(\frac{\alpha_2}{\varepsilon} e^c\right)^2 (|\delta n|_2 + |\delta p|_2)^2. \end{aligned}$$

$$\begin{aligned} \langle (\hat{n} - n)\,\nabla U, \nabla\,\delta n\rangle &\leqslant |\hat{n} - n|_4\,|\nabla U|_4\,|\nabla\,\delta n|_2 \\ &\leqslant \frac{1}{4}|\nabla\,\delta n|_2^2 + |\hat{n} - n|_4^2\,|U|_{W^{1,4}}^2 \end{aligned}$$

$$\begin{aligned} \langle F, \delta n\rangle &\leqslant \bar{Q}e^c(|\delta p|_2 + |\delta n|_2)\,|\delta n|_2 + \bar{Q}\,|P|_2\,|\hat{n} - n|_4\,|\delta n|_4 \\ &\quad + \bar{Q}(e^{2c} + 1)(|\delta n|_2 + |\delta p|_2)\,|\delta n|_2 \\ &\quad + \bar{Q}e^c(|\hat{n} - n|_4 + |\hat{p} - p|_4)(|N|_2 + |P|_2)\,|\delta n|_4 \\ &\leqslant \bar{Q}(e^c + 1)^2\,(|\delta p|_2 + |\delta n|_2)\,|\delta n|_2 \\ &\quad + \frac{\mu_1}{4}|\nabla\,\delta n|_2^2 + \frac{4\alpha_4^2}{\mu_1}\bar{Q}^2(1 + e^c)^2\,(|\hat{n} - n|_4^2 \\ &\quad + |\hat{p} - p|_4^2)(|N|_2^2 + |P|_2^2) \end{aligned}$$

(By (1.5) for $|\delta n|_4$.) Similar estimates for terms involving δp can also be established. Thus from (3.16)–(3.17) we can obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|\delta n|_2^2+|\delta p|_2^2)+\frac{1}{4}(\mu_1\,|\nabla\,\delta n|_2^2+\mu_2\,|\nabla\,\delta p|_2^2)\\
&\quad\leqslant K_1(|\delta n|_2^2+|\delta p|_2^2)+K_2(|\hat{n}-n|_4^2+|\hat{p}-p|_4^2)(|N|_2^2+|P|_2^2+|U|_{W^{1,4}}^2)\\
&\quad\leqslant K_1(|\delta n|_2^2+|\delta p|_2^2)+K_3(|\hat{n}-n|_4^2+|\hat{p}-p|_4^2)(|\hat{n}_0-n_0|_2^2+|\hat{p}_0-p_0|_2^2)
\end{aligned}$$

where we have used (3.8) and (3.14). Since $\delta n(0)=\delta p(0)=0$, by the Gronwall inequality,

$$\begin{aligned}
&|\delta n(t)|_2^2+|\delta p(t)|_2^2\\
&\quad\leqslant 2K_3\int_0^t(|\hat{n}-n|_4^2+|\hat{p}-p|_4^2)\,ds(|\hat{n}_0-n_0|_2^2+|\hat{p}_0-p_0|_2^2). \qquad (3.19)
\end{aligned}$$

From the continuity of the semigroup (see Lemma 2.1),

$$|\hat{n}-n|_4^2\leqslant|\hat{n}-n|_\infty\,|\hat{n}-n|_2\leqslant 2e^c\,|\hat{n}-n|_2\leqslant 2e^ce^{Mt}(|\hat{n}_0-n_0|_2^2+|\hat{p}_0-p_0|_2^2)^{1/2}.$$

Hence (3.19) becomes

$$|\delta n(t)|_2^2+|\delta p(t)|_2^2\leqslant\frac{4K_3}{M}e^c(e^{Mt}-1)(|\hat{n}_0-n_0|_2^2+|\hat{p}_0-p_0|_2^2)^{3/2},$$

and (3.15) follows. Thus we have completed the proof of this theorem. ∎

4. Upper Bound of the Hausdorff Dimension of the Attractor

In this section, we show that the Hausdorff dimension of the attractor $\mathscr{A}$ is finite, and give an upper bound for the dimension of $\mathscr{A}$.

We first assume the asymptotic property of the eigenvalues of the Laplace operator with the mixed boundary conditions:

(H8) For eigenvalues $\{\lambda_j\}_{j=1}^\infty$ of the Laplace operator $-\Delta w$ in $L^2(\Omega)$ with boundary conditions

$$w=0 \text{ on } \Sigma_D \qquad\text{and}\qquad \frac{\partial w}{\partial v}=0 \text{ on } \Sigma_N$$

have the following asymptotic property:

$$\lambda_j\sim cj^{2/l} \qquad \text{as} \quad j\to\infty$$

where c is a constant depending on Ω, and l is the space dimension.

Again, in the one-dimensional case this assumption always holds, and in some two and three-dimensional cases it is also satisfied. For instance, when Ω is the rectangle $(0, a_1) \times (0, a_2) \subset R^2$ and Dirichlet conditions are given on $x_1 = 0$ and $x_1 = a_1$, the eigenvalues are $\lambda_{mn} = (m\pi/a_1)^2 + (n\pi/a_2)^2$ ($m = 1, 2, \ldots$, and $n = 0, 1, 2, \ldots$). In general, when the boundary conditions are not mixed (i.e., either all Nuemann or all Dirichlet throughout the boundary), this asymptotic property is well known (see, e.g., [2]).

We follow the procedure in [13, Section 5.2.3] to establish an upper bound for the dimension of the attractor $\mathscr{A}$. The key is to estimate the number m of the m-dimensional volume element in the phase space so that it decays exponentially.

Suppose $z_0 = (n_0, p_0) \in \mathscr{A}$ and $z(t) = S(t)\, z_0$. Let ξ_j ($j = 1, 2, \ldots, m$) be m elements in $\mathscr{H}$ and $Z_j(t) = L(t; z_0)\, \xi_j$. Let $Q_m = Q_m(t, z_0; \xi_1, \ldots, \xi_m)$ be the projector in $\mathscr{H}$ onto the space spanned by $\{Z_1(t), Z_2(t), \ldots, Z_m(t)\}$. Then the m-dimensional volume element decays exponentially if

$$q_m(t) \equiv -\sup_{z_0 \in \mathscr{A}} \sup_{\{\xi_j\}_1^m} \frac{1}{t} \int_0^t \operatorname{Tr} \sigma(Q_m \cdot, \cdot)\, d\tau \leqslant -\delta < 0$$

for all t large (see [13, Proposition 5.2.1]), where the second sup is taken for all $\{\xi_j\}_1^m \subset \mathscr{H}$ with $|\xi_j|_{\mathscr{H}} \leqslant 1$, $j = 1, \ldots, m$.

Suppose $\{\eta_j(\tau)\}_{j=1}^{\infty}$ is an orthonormal basis of $\mathscr{H}$ at the given time τ, so that

$$\{\eta_1(\tau), \ldots, \eta_m(\tau)\} \text{ spans } Q_m \mathscr{H},$$

the space spanned by $\{Z_1(\tau), \ldots, Z_m(\tau)\}$. Since $Z_j(\tau) \in \mathscr{V}$ from Theorem 3.1, we assume $\eta_j(\tau) \in \mathscr{V}$ ($j = 1, \ldots, m$). Then from the coercivity (3.11) of σ, we have, at time τ,

$$\begin{aligned}
\operatorname{Tr} \sigma(Q_m \cdot, \cdot) &= \sum_{j=1}^{\infty} \sigma(Q_m \eta_j(\tau), \eta_j(\tau)) = \sum_{j=1}^{m} \sigma(\eta_j(\tau), \eta_j(\tau)) \\
&\geqslant \sum_{j=1}^{m} \alpha\, |\eta_j(\tau)|_{\mathscr{V}}^2 - \beta\, |\eta_j(\tau)|_{\mathscr{H}}^2 \\
&= \alpha \sum_{j=1}^{m} |\eta_j(\tau)|_{\mathscr{V}}^2 - \beta m.
\end{aligned}$$

For the positive, self-adjoint operator diag$(-\Delta, -\Delta)$ on $\mathscr{H}$, we apply [13, Lemma 6.2.1] to obtain, from (H8),

$$\sum_{j=1}^{m} |\eta_j(\tau)|_{\mathscr{V}}^2 \geqslant c' m^{1+2/l}$$

where c' is independent of η_j, m and τ. Hence

$$\operatorname{Tr} \sigma(Q_m \cdot, \cdot) \geqslant c'\alpha m^{1+2/l} - \beta m$$

where α and β are given in (3.13) with e^c being the upper L^∞ bound for $(n(\tau), p(\tau)) = S(\tau) z_0$. From Corollary 2.6, $S(\tau) z_0$ has the uniform upper L^∞ bound $1 + C$ (C is as in (1.10)) for all $z_0 \in \mathscr{A}$. Therefore,

$$q(t) \leqslant -\frac{c'}{2} \underline{\mu} m^{1+2/l} + \beta_0 m$$

where

$$\beta_0 = \frac{1}{2\varepsilon} \bar{\mu}(C + 1 + |D|_\infty) + \frac{2}{\varepsilon^2} \bar{\mu}(C+1)^2 + 2\bar{Q}(C+2)^2$$

and C is the same constant in the description of the absorbing set (1.12). Hence, if we choose the integer m such that

$$m - 1 \leqslant \left(\frac{2\beta_0}{c'\underline{\mu}} \right)^{l/2} < m, \tag{4.1}$$

then $q_m(t) \leqslant -\delta < 0$ for all $t > 0$, and hence the m-dimensional volume element decays exponentially in the phase space. Furthermore, the Hausdorff dimension of $\mathscr{A}$ is finite and no larger than m ([13, Theorem 5.3.3]).

We summarize the above discussion in the following theorem.

THEOREM 4.1. *Under the hypotheses* (H1)–(H8), *the attractor* $\mathscr{A}$ *of the dynamical system has a finite Hausdorff dimension, and the dimension is less than or equal to the integer* m *given in* (4.1).

An expression for C in terms of the model parameters can be found in [4, Section 4.1]. To obtain a simpler expression for this case of constant mobilities, we can follow the same calculation there. Our calculation yields

$$C = 2C_1 + 2^9 C_1^{-2} (\mu_1^{-1} + \mu_2^{-1})^{3/4} |\Omega|^{3/4} M^{3/2}$$

where

$$C_1 = \sup_{\Sigma_D} \{\bar{n}, \bar{p}\},$$

$$M = \frac{2\alpha_2^2 \alpha_4^2}{\underline{\mu}^{3/2}} \left[\left(\frac{|D|_\infty^2}{4} + C_1 |D|_\infty \right) \frac{\bar{\mu}}{\varepsilon} + \bar{Q} + |g|_\infty + \frac{\bar{\mu}}{4\alpha_2^2} C_1 \right]$$

and α_2 and α_4 are the Poincaré constants in (1.5).

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